Discrete Ill-Posed and Rank-Deficient Problems

Overview

- Definitions
  - Inversion, SVD, Picard Condition,
  - Rank Deficient, Ill-Posed
- “Classical” Regularization
  - Tikhonov and the Discrete Smoothing Norms
- Discretization
  - Quadrature and Galerkin

Overview 2

- Discretization
  - Discrete Picard Condition
  - Quadrature
  - Galerkin
- Other Techniques
  - TSVD
  - CG Interation

Inversion

$$Ax = b$$

- Solve for x, when A and b are known
  - (or for A, when x and b are known)
  - via Inverse, Psuedo-Inverse

$$x = A^{-1} b \qquad x = A^{t} b$$

SVD

- SVD(A)
  - or GSVD(A,L)

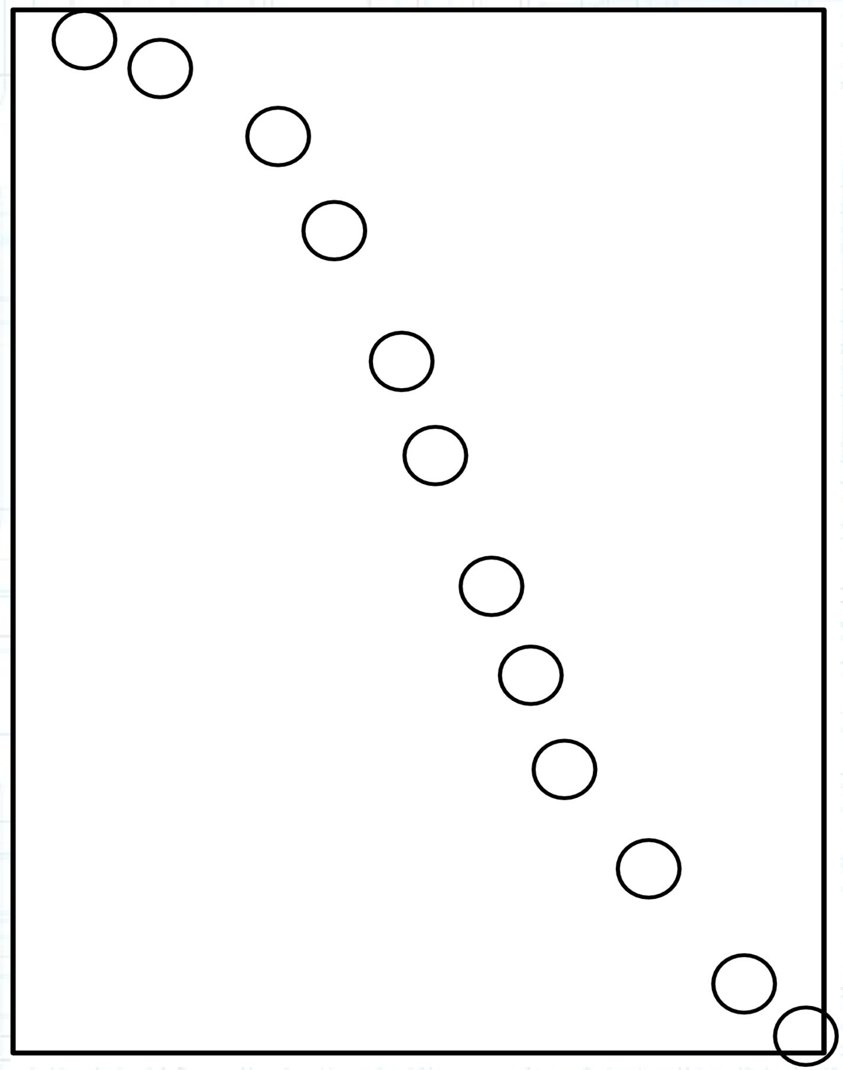
$$A = U \Sigma V^T = \sum_{i=1}^{n} u_i \sigma_i v_i^T$$

$$U^T U = V V^T = I_n$$

$$A^T A = V \Sigma^2 V^T \qquad A A^T = U \Sigma^2 U^T$$

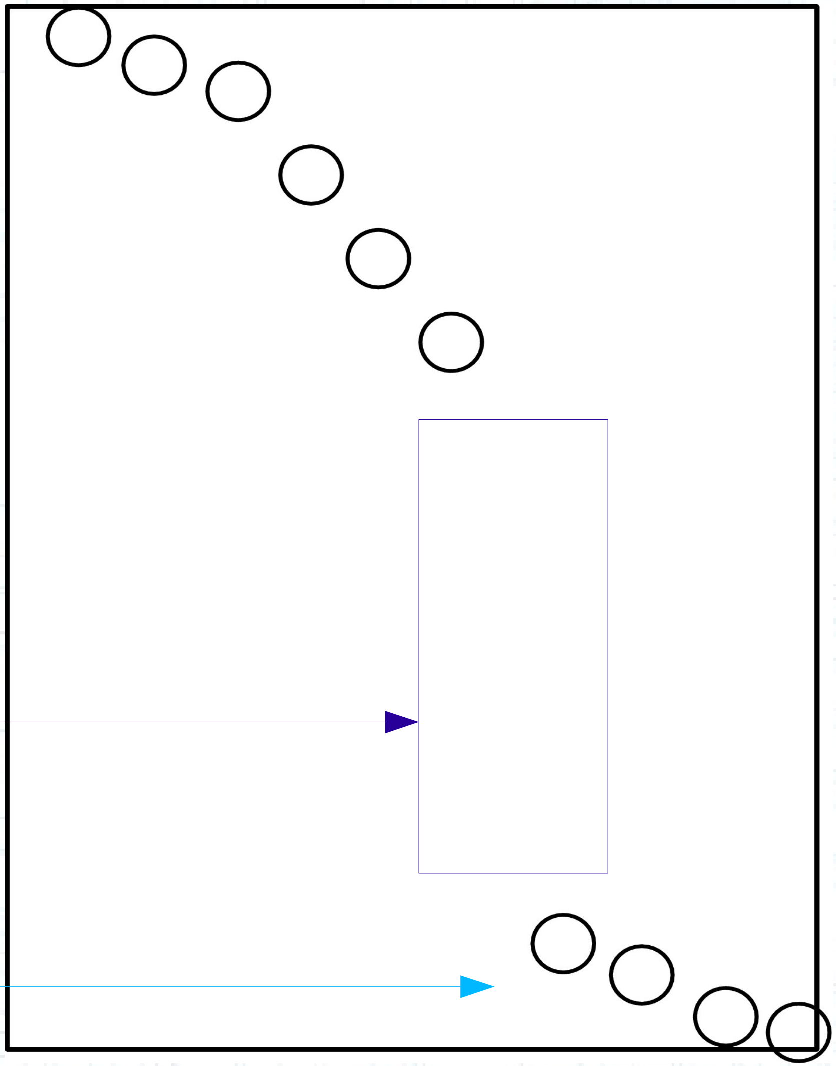
- Singular Values (2)
- Singular Vectors
  - Left & Right

σ in non-increasing order



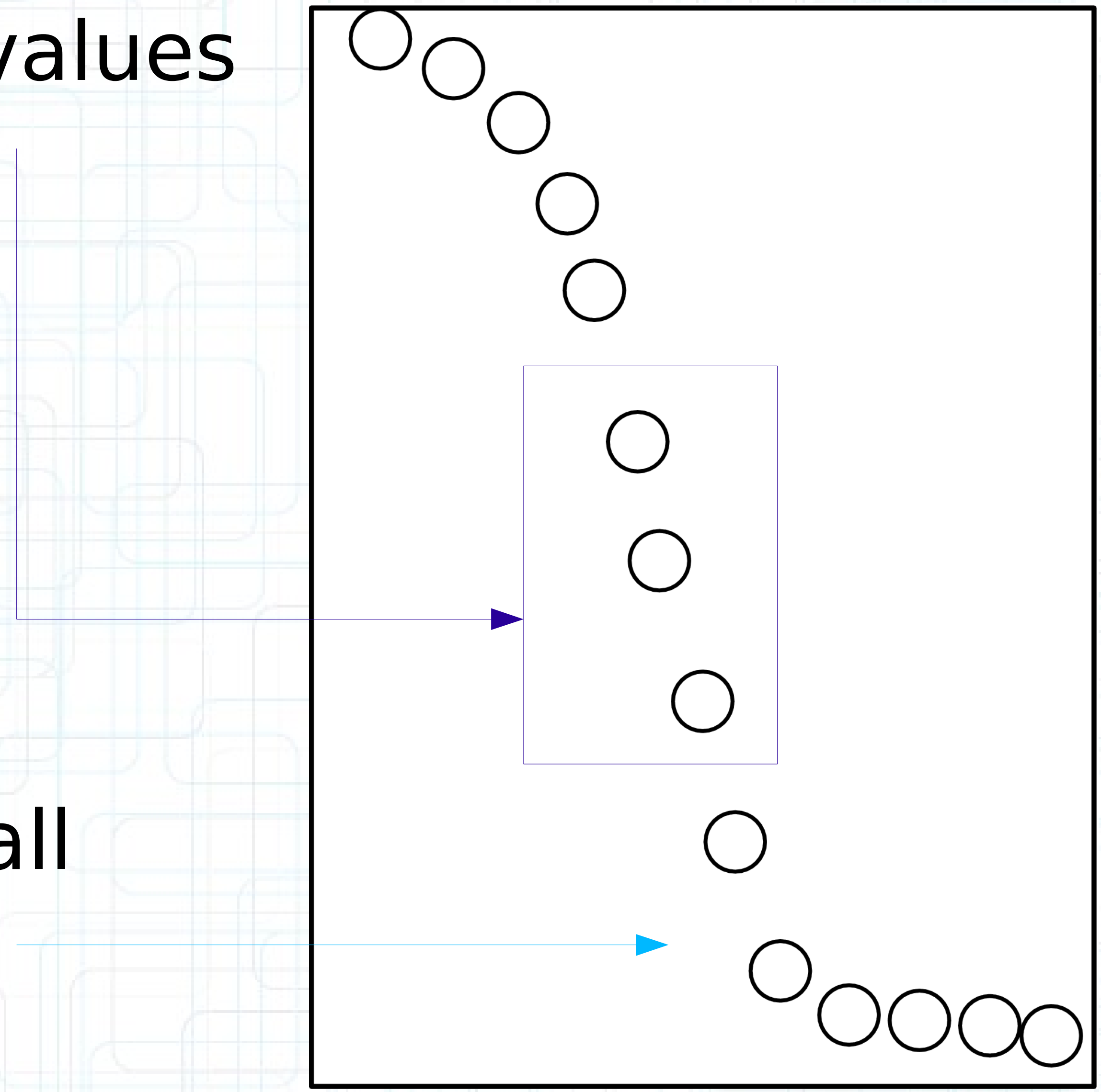
Rank Deficient

- Gap in the Singular Values
- Approx. Zero
  - Noise!



Ill-Posed

- No gap in values
- Lots of small values



(Discrete) Picard Condition

Continuous

$$\int_0^1 K(s,t)\,f(t) = g(s)$$

SVE

$$K(s,t) = \sum_{i=1}^{\infty} \mu_i u_i(s) v_i(t)$$

$$f(t) = \sum_{i=1}^{\infty} \frac{(u_i, g)}{\mu_i} v_i(t)$$

$$\sum_{i=1}^{\infty} \left(\frac{(u_i, g)}{\mu_i} \right)^2 < \infty$$

Discrete

$$Ax = b$$

SVD

$$A = U \Sigma V^T = \sum_{i=1}^{n} u_i \sigma_i v_i^T$$

$$x = A^t b = \sum_{i=1}^{n} \frac{u_i^T b}{\sigma_i} v_i$$

...

denominator must decrease faster than numerator or we get an unbounded solution

Regularization

- "First regularize, then discretize."

- Minimize "residual norm" $r = \int_0^1 K(s,t) f(t) - g(s)$
  - with Constrained Values, $min \|r\|_2^2 , \quad x \in S$
  - with Constrained Size, $min \|r\|_2^2 , \quad w() < \delta$
- Minimize Size, with Constrained Residual $min \|w()\|_2^2 , \quad \|r\|_2^2 < \delta$
- Minimize Residual and Size
  - Tikhonov! $min \{ \|r\|_2^2 + \lambda^2 \|w()\|_2^2 \}$

Tikhonov Regularization

$$min\{ \|Ax-b\|_2^2 \;+\; \lambda^2\|L(x-x_0)\|_2^2 \}$$

- Solved by Least Squares using SVD

Discrete Smoothing Norms

- Choices for “L”:
  - Identity Matrix
  - Weighted Diagonal
  - Discrete Derivative Approximations

L = I

- The identity matrix...

Additional constraint:

  - Minimize the absolute value of the solution

$$min\{ \|Ax-b\|_2^2 + \lambda^2 \|I(x-x_0)\|_2^2 \}$$

L = diag(w)

- A diagonal matrix of weights on x...

Additional constraint:

  - Minimize a weighted selection of values of the solution

$$min\{\ \|Ax-b\|_2^2\ +\ \lambda^2\|W(x-x_0)\|_2^2\ \}$$

$$W=diag(w)$$

L = L1 or L2

- A banded matrix approximating a derivative operator...

Additional constraint:

  - Minimize "total variation" in values of the solution (but still allow steep gradients)

$$min\{ \|Ax-b\|_2^2 + \lambda^2 \|L_1(x-x_0)\|_2^2 \}$$

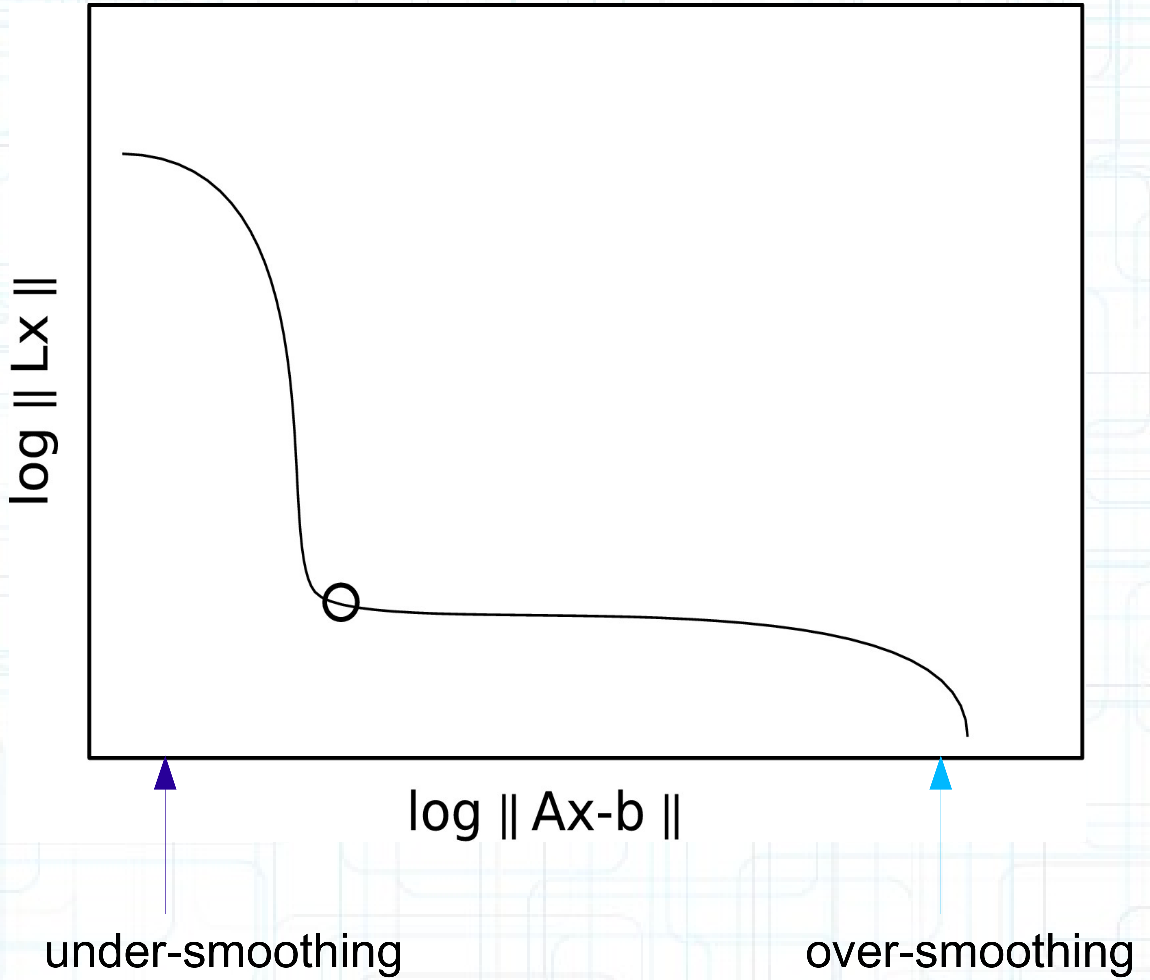
Priors

- Allow the addition of *"a priori"* information about the result

$$min\{ \|Ax-b\|_2^2 \; + \; \lambda^2 \|L(x-x_0)\|_2^2 \}$$

- Using no prior is the same as a zero prior
- Weight solution towards expectation

L-curve



(fake plot)

Discretization of Integral Equations

- Quadrature

$$choose\ w_j$$

$$\int_0^1 \phi(t)\,dt \simeq \sum_{j=1}^{n} w_j \phi(t_j) \qquad a_{ij} = w_j K(s,t) \qquad b_i = g(s_i)$$

- Galerkin

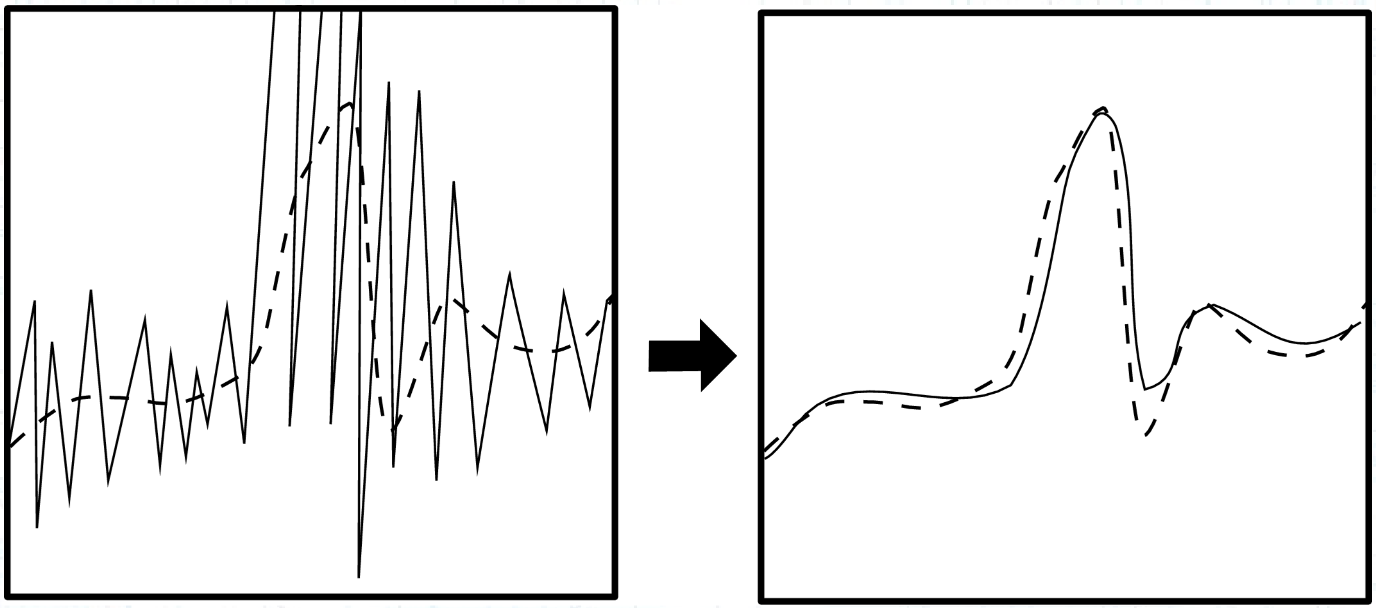
$$choose\ \phi, \psi$$

$$a_{ij} = \int_0^1 K(s,t)\,\phi_i(s)\,\psi_j(t)\,ds\,dt \qquad b_i = \int_0^1 g(s)\,\phi_i(s)\,ds$$

  - Raliegh-Ritz

If $\phi = \psi$, K is symetric, and nodes are co-located

Solution



(fake plots)

Discussion

[http://www.clipartguide.com/_pages/0808-0712-3117-5830.html]

References

- P. C. Hansen, Rank-Deficient and Discrete Ill-Posed Problems, SIAM, 1998
- Wikipedia: Inverse Problems, http://en.wikipedia.org/wiki/Inverse_problem, visited Jan 28, 2009
- J Kaipio, E Somersalo, Statistical and Computational Inverse Problems, Springer, 2005