Discrete Ill-Posed and Rank-Deficient Problems
Overview

- Definitions
  - Inversion, SVD, Picard Condition,
  - Rank Deficient, Ill-Posed
- “Classical” Regularization
  - Tikhonov and the Discrete Smoothing Norms
- Discretization
  - Quadrature and Galerkin
Overview 2

• Discretization
  – Discrete Picard Condition
  – Quadrature
  – Galerkin

• Other Techniques
  – TSVD
  – CG Iteration
Inversion

\[ Ax = b \]

- Solve for \( x \), when \( A \) and \( b \) are known
  - (or for \( A \), when \( x \) and \( b \) are known)
  - via Inverse, Pseudo-Inverse

\[ x = A^{-1} b \quad x = A^t b \]
SVD

- **SVD(A)**
  - or GSVD(A,L)

\[ A = U \Sigma V^T = \sum_{i=1}^{n} u_i \sigma_i v_i^T \]

\[ U^T U = V V^T = I_n \]

\[ A^T A = V \Sigma^2 V^T \]
\[ A A^T = U \Sigma^2 U^T \]

- **Singular Values**
- **Singular Vectors**
  - Left & Right
Rank Deficient

- Gap in the Singular Values
- Approx. Zero
  - Noise!
Ill-Posed

- No gap in values
- Lots of small values
(Discrete) Picard Condition

Continuous

\[ \int_0^1 K(s, t) f(t) = g(s) \]

\[ K(s, t) = \sum_{i=1}^{\infty} \mu_i u_i(s) v_i(t) \]

\[ f(t) = \sum_{i=1}^{\infty} \frac{(u_i, g)}{\mu_i} v_i(t) \]

\[ \sum_{i=1}^{\infty} \left( \frac{(u_i, g)}{\mu_i} \right)^2 < \infty \]

Discrete

\[ A x = b \]

\[ A = U \Sigma V^T = \sum_{i=1}^{n} u_i \sigma_i v_i^T \]

\[ x = A^T b = \sum_{i=1}^{n} \frac{u_i^T b}{\sigma_i} v_i \]

\[ \ldots \]

denominator must decrease faster than numerator or we get an unbounded solution
Regularization

- “First regularize, then discretize.”

- Minimize “residual norm”
  - with Constrained Values, $\min \|r\|_2^2, \ x \in S$
  - with Constrained Size, $\min \|r\|_2^2, \ w() < \delta$

- Minimize Size, with Constrained Residual
  $\min \|w()\|_2^2, \ ||r||_2^2 < \delta$

- Minimize Residual and Size
  $\min \{||r||_2^2 + \lambda^2 ||w()||_2^2\}$

\[ r = \int_0^1 K(s, t) f(t) - g(s) \]
Tikhonov Regularization

\[
\min \left\{ \|Ax - b\|_2^2 + \lambda^2 \|L(x - x_0)\|_2^2 \right\}
\]

- Solved by Least Squares using SVD
Discrete Smoothing Norms

• Choices for “L”:
  - Identity Matrix
  - Weighted Diagonal
  - Discrete Derivative Approximations
The identity matrix...

Additional constraint:

- Minimize the absolute value of the solution

\[
\min \left\{ \| Ax - b \|_2^2 + \lambda^2 \| I(x - x_0) \|_2^2 \right\}
\]
L = diag(w)

- A diagonal matrix of weights on x...

Additional constraint:

- Minimize a weighted selection of values of the solution

\[
\min \left\{ \|Ax - b\|_2^2 + \lambda^2 \|W(x - x_0)\|_2^2 \right\}
\]

\[
W = \text{diag} \left( w \right)
\]
**L = L1 or L2**

- A banded matrix approximating a derivative operator...

Additional constraint:
- Minimize “total variation” in values of the solution (but still allow steep gradients)

\[
\min \left\{ \|Ax - b\|_2^2 + \lambda^2 \|L_1(x - x_0)\|_2^2 \right\}
\]
Priors

- Allow the addition of “a priori” information about the result

\[
\min \{ \|Ax - b\|_2^2 + \lambda^2 \|L(x - x_0)\|_2^2 \}
\]

- Using no prior is the same as a zero prior

- Weight solution towards expectation
L-curve

(fake plot)

\[ \log \| Lx \| \quad \log \| Ax - b \| \]

under-smoothing   over-smoothing

(fake plot)
Discretization of Integral Equations

- **Quadrature**
  
  \[
  \int_0^1 \phi(t) \, dt \approx \sum_{j=1}^n w_j \phi(t_j)
  \]
  
  \[
  a_{ij} = w_j K(s, t) \quad b_i = g(s_i)
  \]

  \[\text{choose } w_j\]

- **Galerkin**
  
  \[
  a_{ij} = \int_0^1 K(s, t) \phi_i(s) \psi_j(t) \, ds \, dt
  \]
  
  \[
  b_i = \int_0^1 g(s) \phi_i(s) \, ds
  \]

  \[\text{choose } \phi, \psi\]

- **Raliegh-Ritz**

  If \( \phi = \psi \), \( K \) is symmetric, and nodes are co-located
Solution

(fake plots)
Discussion

References