Meet "the Big Time"

Spatio-Temporal Regularization over Many Frames

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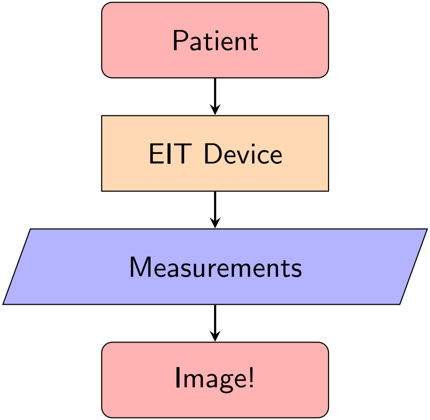
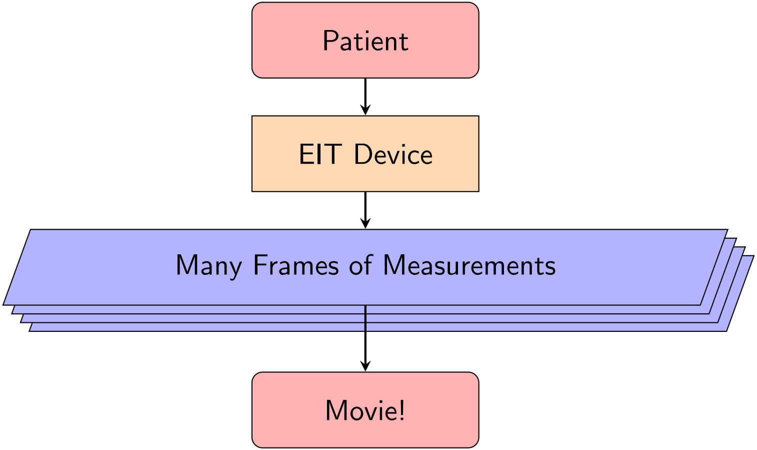
Imaging

a single frame of data



Monitoring

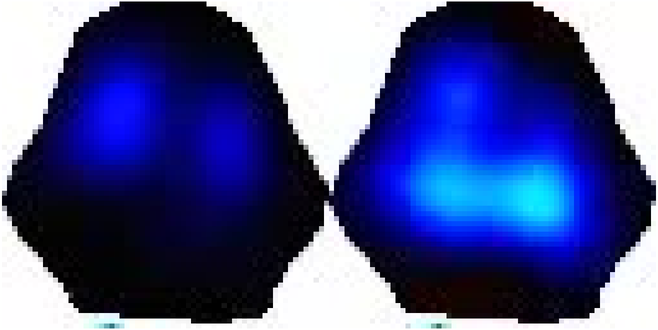
30 frames per second



This requires heavier spatial regularization to get “smooth” frame-to-frame transitions, as much as possible.
Is our movie over regularized?

Example: A PEEP Trial

Pig incremental/decremental PEEP trial[1]



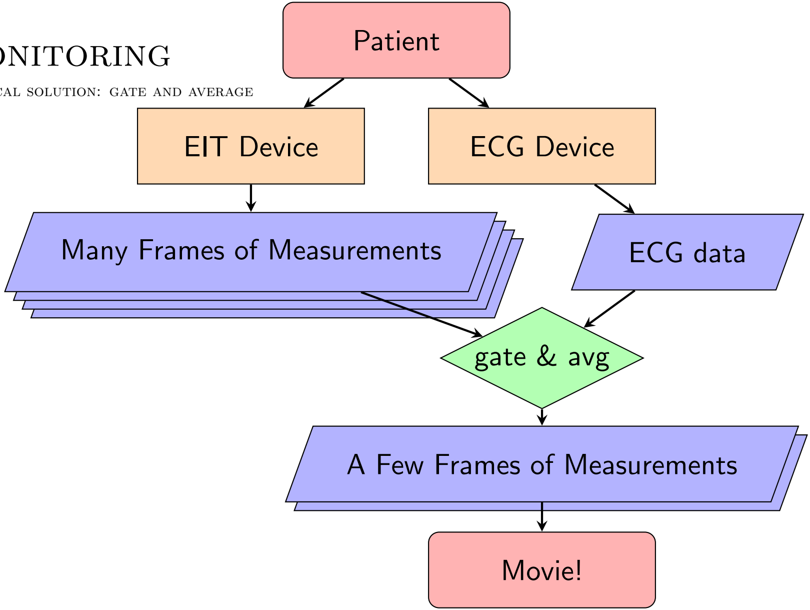
5000 frames @ 30 frames per second; GREIT[2] on each frame

[1] I. Frerichs, P. A. Dargaville, T. Dudykevych, *et al.*, "Electrical impedance tomography: A method for monitoring regional lung aeration and tidal volume distribution?" *Intensive Care Medicine*, vol. 29, no. 12, pp. 2312–2316, Dec. 2003.

[2] A. Adler, J. Arnold, R. Bayford, *et al.*, "Greit: A unified approach to 2D linear EIT reconstruction of lung images," *Physiological Measurement*, vol. 30, no. 6, S35–S55, Jun. 2009.

Monitoring

a typical solution: gate and average



Gauss-Newton: Spatial

single-step or iterative

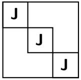
$$\mathbf{x}_{n+1} = (\mathbf{J}^\mathsf{T}\mathbf{W}\mathbf{J} + \lambda\mathbf{R}^\mathsf{T}\mathbf{R})^{-1}\left(\mathbf{J}^\mathsf{T}\mathbf{W}\mathbf{b} + \lambda\mathbf{R}^\mathsf{T}\mathbf{R}(\mathbf{x}_* - \mathbf{x}_n)\right)$$

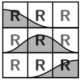
$\mathbf{J}$ – Jacobian
$\mathbf{W}$ – inv noise cov
λ – hyperparameter
$\mathbf{R}$ – spatial regularization

$\mathbf{b}$ – measurements
$\mathbf{x}_*$ – prior estimate
$\mathbf{x}_0$ – initial guess
$\mathbf{x}_n$ – last estimate
$\mathbf{x}_{n+1}$ – next estimate

Gauss-Newton: Spatio-Temporal

by block-wise expansion

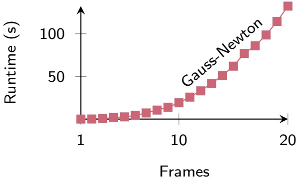
$\mathbf{I} \otimes \mathbf{J} =$ 

$\boldsymbol{\Gamma} \otimes \mathbf{R} =$ 

The spatial Gauss-Newton update becomes a
spatio-temporal Gauss-Newton update[3]

$$\mathrm{vec}(\mathbf{X}_{n+1}) = \left((\mathbf{I} \otimes \mathbf{J})^{\mathsf{T}}(\mathbf{I} \otimes \mathbf{W})(\mathbf{I} \otimes \mathbf{J}) + \lambda(\boldsymbol{\Gamma} \otimes \mathbf{R})^{\mathsf{T}}(\boldsymbol{\Gamma} \otimes \mathbf{R})\right)^{-1}$$
$$\left((\mathbf{I} \otimes \mathbf{J})^{\mathsf{T}}(\mathbf{I} \otimes \mathbf{W})\mathrm{vec}(\mathbf{B}) + \lambda(\boldsymbol{\Gamma} \otimes \mathbf{R})^{\mathsf{T}}(\boldsymbol{\Gamma} \otimes \mathbf{R})\mathrm{vec}(\mathbf{X}_* - \mathbf{X}_n)\right)$$

... can be simplified a bit but still has the same storage requirements



exponential memory storage and runtime growth as # frames, squared

[3] T. Dai, M. Soleimani, and A. Adler, "EIT image reconstruction with four dimensional regularization," *Medical & Biological Engineering & Computing*, vol. 46, no. 9, pp. 889–899, 2008.

Kronecker product identity

identity:

$$\text{vec}(\mathbf{AXB}) = \text{vec}(\mathbf{C}) = (\mathbf{B}^\mathsf{T} \otimes \mathbf{A})\text{vec}(\mathbf{X})$$

which transforms our spatio-temporal Gauss-Newton update into

$$\text{vec}(\mathbf{J}^\mathsf{T}\mathbf{W}\mathbf{J}\mathbf{X}_{n+1} + \lambda\mathbf{R}^\mathsf{T}\mathbf{R}\mathbf{X}_{n+1}\mathbf{\Gamma}\mathbf{\Gamma}^\mathsf{T}) = \text{vec}\left(\mathbf{J}^\mathsf{T}\mathbf{W}\mathbf{B} + \lambda\mathbf{R}^\mathsf{T}\mathbf{R}(\mathbf{X}_* - \mathbf{X}_n)\mathbf{\Gamma}\mathbf{\Gamma}^\mathsf{T}\right)$$

Note that we've removed all the Kronecker products that blew up our matrices, but we now have $\mathbf{X}_{n+1}$ in the middle of the equation

Inner Conjugate Gradient

$$\text{vec}(\mathbf{J}^\mathsf{T}\mathbf{W}\mathbf{J}\mathbf{X}_{n+1} + \lambda\mathbf{R}^\mathsf{T}\mathbf{R}\mathbf{X}_{n+1}\boldsymbol{\Gamma}\boldsymbol{\Gamma}^\mathsf{T}) = \text{vec}\left(\mathbf{J}^\mathsf{T}\mathbf{W}\mathbf{B} + \lambda\mathbf{R}^\mathsf{T}\mathbf{R}(\mathbf{X}_* - \mathbf{X}_n)\boldsymbol{\Gamma}\boldsymbol{\Gamma}^\mathsf{T}\right)$$

Use an iterative Conjugate Gradient[4] solution in place of direct left-divide (LU decomposition).

- left-hand side computed on-the-fly at each inner CG iteration
- right-hand side computed once at each outer GN iteration
- each side should be computed to minimize matrix sizes, maximize sparsity

... still slow

until we limit the number of iterations, watch convergence and adjust stopping criteria, which could be done rigourously[5]

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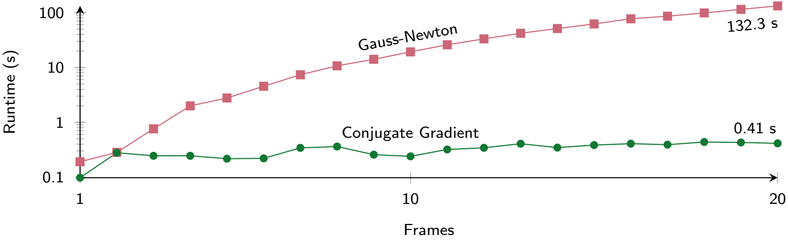
[4] J. Shewchuk, "An introduction to the conjugate gradient method without the agonizing pain," Carnegie Mellon University, Tech. Rep., 1994.

[5] A. Rieder, "Inexact newton regularization using conjugate gradients as inner iteration," *SIAM Journal on Numerical Analysis*, vol. 43, no. 2, pp. 604–622, 2006.

Better

and numerically equivalent



Napkin View

- what: monitoring approaches lead to time series data,
  lots of data @ 30 f/s!
- why: do we filter data frames over time (ala FBP for spatial)
  or
  "Can we do better?"
- how: regularize over space *and* time... Spatio-Temporal

- but: math by boxes... it gets too big!
- how: a Kronecker product identity that almost works
- how: "But we don't do that!" .. Conjugate Gradients
- bonus: cyclical events ... ECG-gated data
- bonus: regional spatio-temporal regularization by block-wise matrices