Spatio-Temporal Regularization over Many Frames

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Abstract: Regularizing over both spatial and temporal spaces for EIT data can lead to very large matrices which can be challenging to compute. The Kronecker product identity may be leveraged with the Conjugate Gradient method to construct a system of equations that scales linearly with the number of data frames collected and reconstruction parameters.

1 Introduction

Gauss-Newton methods are generally used to reconstruct an Electrical Impedance Tomography (EIT) conductivity image for a single frame of data. Multiple frames of data may be reconstructed together and have regularization applied across them, leading to spatio-temporal regularization.

2 Gauss-Newton

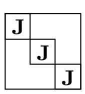
The Gauss-Newton iterative update (GN-update) is

$$\mathbf{x}_{n+1} = (\mathbf{J}_2 + \lambda \mathbf{R}_2)^{-1} \left(\mathbf{J}^\intercal \mathbf{W} \mathbf{b} + \lambda \mathbf{R}_2 (\mathbf{x}_* - \mathbf{x}_n) \right) \quad (1)$$

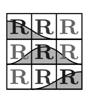
where $\mathbf{J}_2 = \mathbf{J}^\intercal \mathbf{W} \mathbf{J}$ and $\mathbf{R}_2 = \mathbf{R}^\intercal \mathbf{R}$. New parameters $\mathbf{x}_{n+1}$ are calculated using results from iteration n and with a prior estimate $\mathbf{x}_*$, where the Jacobian $\mathbf{J}$ is calculated based on $\mathbf{x}_n$, the measurements are weighted by an inverse noise covariance matrix $\mathbf{W}$, and the reconstruction is regularized by $\mathbf{R}$ with a hyperparameter λ controlling regularization strength.

Following [1], the GN-update (1) may be expanded as a block-diagonal matrix to handle many frames in a time-series of data and reconstruct these frames simultaneously while applying regularization across time. Entries in the time series are assigned an exponential smoothing $\boldsymbol{\Gamma}$, so that adjacent frames are assumed to be strongly correlated. The same spatial regularization $\mathbf{R}$ is applied to every frame's reconstruction.

$$\mathbf{I} \otimes \mathbf{J} =$$



$$\boldsymbol{\Gamma} \otimes \mathbf{R} = \quad (2)$$



The GN-update becomes

$$\begin{aligned} \mathrm{vec}(\mathbf{X}_{n+1}) = \big(&(\mathbf{I}\otimes\mathbf{J})^\intercal (\mathbf{I}\otimes\mathbf{W})(\mathbf{I}\otimes\mathbf{J}) + \\ &\lambda (\boldsymbol{\Gamma}\otimes\mathbf{R})^\intercal (\boldsymbol{\Gamma}\otimes\mathbf{R}) \big)^{-1} \\ &\big((\mathbf{I}\otimes\mathbf{J})^\intercal (\mathbf{I}\otimes\mathbf{W}) \mathrm{vec}(\mathbf{B}) + \\ &\lambda (\boldsymbol{\Gamma}\otimes\mathbf{R})^\intercal (\boldsymbol{\Gamma}\otimes\mathbf{R}) \mathrm{vec}(\mathbf{X}_* - \mathbf{X}_n) \big) \end{aligned} \quad (3)$$

where $\mathbf{X}$ denotes the reconstruction parameters joined into a matrix with one column per frame and the vec() operator reshapes this matrix into a single column vector. The measurements $\mathbf{B}$ are treated similarly. Using Kronecker product identities, (3) may be written

$$\begin{aligned} \mathrm{vec}(\mathbf{X}_{n+1}) &= (\mathbf{I}\otimes\mathbf{J}_2 + \lambda \boldsymbol{\Gamma}_2 \otimes \mathbf{R}_2)^{-1} \\ &\big(\mathbf{I} \otimes (\mathbf{J}^\intercal \mathbf{W}) \mathrm{vec}(\mathbf{B}) + \lambda \boldsymbol{\Gamma}_2 \otimes \mathbf{R}_2 \mathrm{vec}(\mathbf{X}_* - \mathbf{X}_n) \big) \end{aligned} \quad (4)$$

where $\boldsymbol{\Gamma}_2 = \boldsymbol{\Gamma}^\intercal \boldsymbol{\Gamma}$. This formulation can result in very large, dense matrices $\mathbf{I} \otimes \mathbf{J}_2$. The Wiener filter form, as suggested in [1], may help though it still gives the large dense matrices.

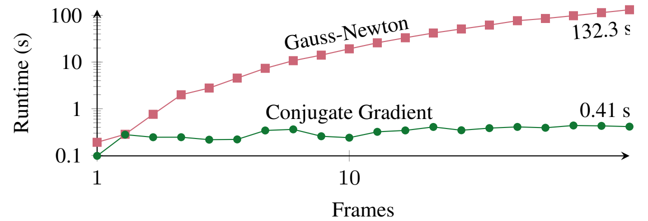


Figure 1: Runtime for a 16 electrode 2D Finite Element mesh (1600 elements) for 1 to 20 frames of 208 measurements using Gauss Newton (GN) (4) and Conjugate Gradient (CG) (7); CG scales to many frames of measurement data while GN runs exponentially slower as more frames are added

3 Conjugate Gradient

The Conjugate Gradient (CG) update [2] efficiently calculates the inverse in (4) by iterative evaluation of

$$\begin{aligned} &(\mathbf{I}\otimes\mathbf{J}_2 + \lambda\boldsymbol{\Gamma}_2\otimes\mathbf{R}_2)\mathrm{vec}(\mathbf{X}_{n+1}) = \\ &\quad \mathbf{I}\otimes(\mathbf{J}^\intercal\mathbf{W})\mathrm{vec}(\mathbf{B}) + \lambda\boldsymbol{\Gamma}_2\otimes\mathbf{R}_2\mathrm{vec}(\mathbf{X}_* - \mathbf{X}_n) \end{aligned} \quad (5)$$

A key identity of the Kronecker product may be used to significantly reduce the computational requirements

$$\mathrm{vec}(\mathbf{AXB}) = \mathrm{vec}(\mathbf{C}) = (\mathbf{B}^\intercal \otimes \mathbf{A})\mathrm{vec}(\mathbf{X}) \quad (6)$$

which transforms (5) into

$$\begin{aligned} &\mathrm{vec}(\mathbf{J}_2\mathbf{X}_{n+1} + \lambda\mathbf{R}_2\mathbf{X}_{n+1}\boldsymbol{\Gamma}_2^\intercal) = \\ &\quad \mathrm{vec}\left(\mathbf{J}^\intercal\mathbf{W}\mathbf{B} + \lambda\mathbf{R}_2(\mathbf{X}_* - \mathbf{X}_n)\boldsymbol{\Gamma}_2^\intercal \right) \end{aligned} \quad (7)$$

where, by judicious choice in the order of operations, one can maintain a minimal storage footprint. Note that in the GN solution, solving (5) would result in the same very large matrices as (4), while for CG the Kronecker products do not need to be expanded.

The CG method typically computes the solution $\mathbf{X}_{n+1}$ to a certain precision. For ill-posed problems, the accuracy of the parametrization is limited by measurement noise and regularization. Stopping the conjugate gradient iterations early avoids getting trapped in fruitless iterations. Rigorously CG stopping criteria for EIT CG-updates have been developed in [3], but were heuristically found in this work through plotting of the CG error estimates. Halting CG iterations when the algorithm started to oscillate gave nearly identical results.

4 Discussion

Spatio-temporal regularization combining the techniques described in this work, the Kronecker product identities and the Conjugate Gradient method, may be brought together to tackle previously uncomputable EIT data sets.

References

[1] Dai T, Soleimani M, et al. *Med Biol Eng Comput* **46**(9):889–899, 2008

[2] Shewchuk J. An introduction to the conjugate gradient method without the agonizing pain. Tech. rep., Carnegie Mellon University, 1994

[3] Rieder A. *SIAM J Numer Anal* **43**(2):604–622, 2006