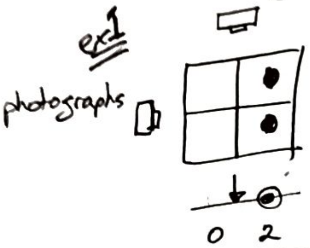
Pictures and CT and, eventually EIT
Electrical Impedance Tomography .

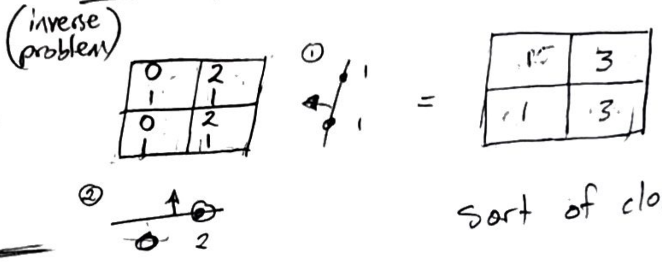
Apr 4, 2018

Projections (forward problem)

Back Projection (inverse problem)

ex1





sort of close.

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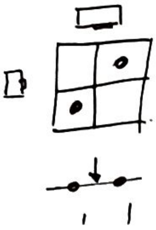
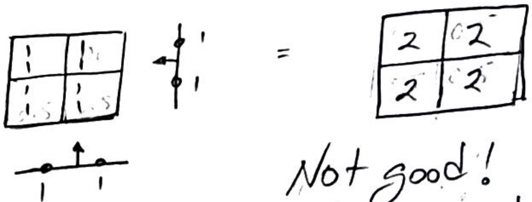
ex2



uhoh.



Not good!
what happened?

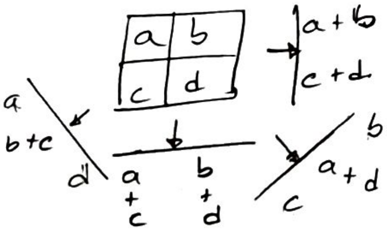
ambiguous measurements

two equally valid solutions

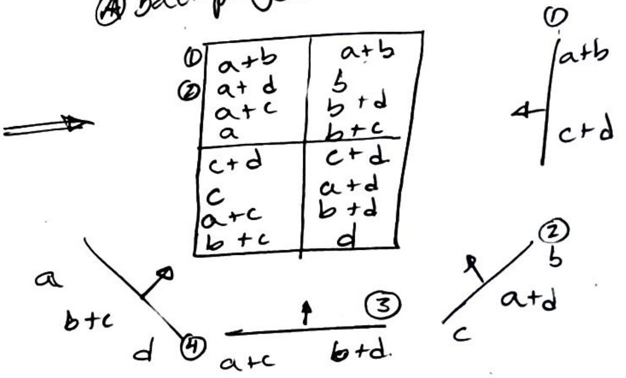
give same measurements.!

How can we solve the ambiguity?

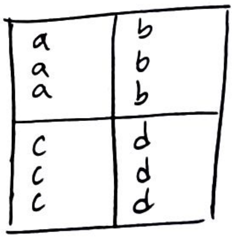
More angles.



Ⓐ backproject



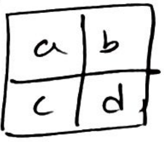
Ⓑ subtract projection sum : -(a+b+c+d)



!

Ⓒ divide by projections minus one.

$\frac{x}{(4-1)}$



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Will MANY more angles make a perfect picture?

No.: in the limit (continuous data), we still have measurement noise // and imperfect reconstruction algorithms.

Error Sources
- discrete data
- measurement noise (statistical errors).
- systematic errors (detector misalignment)
- * - incomplete data
- * - scattering, absorption

Recall ~~For~~ For CT, one approach is filtered back projection (FBP)

FBP

$$f(x,y) = \int_0^{\pi} d\phi \left[\int_{-\infty}^{\infty} d\omega \, |\omega| P(\omega) e^{2\pi j \omega s} \right]_{s = x\cos\phi + y\sin\phi}$$

1) FT of projection $p(s,\phi) \xrightarrow{FT} P(\omega) e^{j 2\pi \omega s}$
2) Multiply by frequency filter : ramp $|\omega|$

- minimize blurring (low freq.)
- maximize contrast (high freq).

3) Inverse FT
4) Back project
5) Sum over all filtered back projections.

→ complete data, no noise + FBP = original 2D image.

Another way: Maximum Liklihood, Expectation Maximization.

or

Gauss - Newton (Iterative)

fwd.
$M = Jx$
$J^T M = (J^T J) x$
$(J^TJ)^{-1} J^T m = (J^TJ)^{-1}(J^TJ)x$
$(J^TJ)^{-1} J^T m = x$ inv

$$\underset{\text{next estimate}}{x_{n+1}} = \underset{\text{last estimate}}{x_n} - \underbrace{(\underset{\text{Jacobian}}{J^T} J)^{-1} J^{-1}}_{\text{pseudo-inverse } (J^{+})} b_n \quad ; \quad \underset{\text{error}}{b_n} = \underset{\text{forward model}}{F(x_n)} - \underset{\text{measurements}}{m} \quad ; \quad \underset{\text{initial guess.}}{x_0}$$

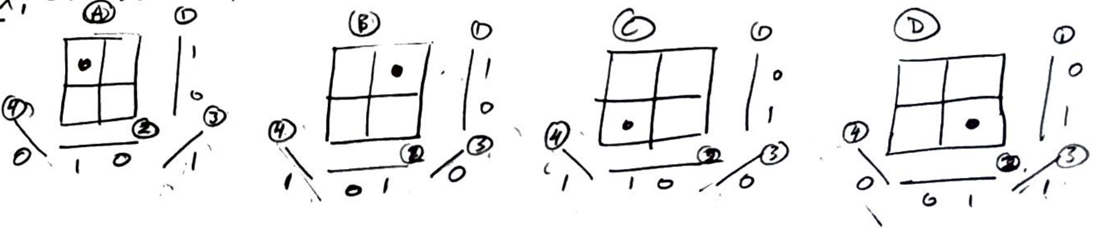
simulate measurements.

... for PET, SPECT, CT, EIT

Jacobian, or sensitivity, from forward model

a "perturbation"

$$J_{ij} = \frac{\partial m_i}{\partial x_j}$$



$$J = \begin{array}{c} \\ \\ \\ \\ \\ \\ \\ \end{array}\begin{array}{cccc|c} Ⓐ & Ⓑ & Ⓒ & Ⓓ & \\ 1 & 1 & 0 & 0 & ① \\ 0 & 0 & 1 & 1 & \\ 0 & 0 & 1 & 0 & ② \\ 0 & 1 & 0 & 1 & \\ 1 & 0 & 0 & 1 & ③ \\ 0 & 1 & 1 & 0 & ④ \end{array}$$

ex $m_A = J x_A = F(x_A)$

for $x_A = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

*Remember linear algebra?

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Try it!

Initial guess : $x_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \emptyset$; Measurements Ⓐ ; solution Ⓐ

"empty"

$$x_1 = x_0 - (J^TJ)^{-1} J^T b_n$$

$$x_1 = x_0 - (J^TJ)^{-1}\left[J^T(Jx_0 - m_A)\right] \quad ; \quad m_A = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = Jx_A$$

$$x_1 = \quad + (J^TJ)^{-1} J^T m_A$$

$$x_1 = \underbrace{(J^TJ)^{-1} J^TJ}_{I} x_A$$

$$x_1 = \boxed{x_A} \longrightarrow \text{next iteration error} = \emptyset \text{, stop!}$$

BUT noise + errors means $m_A \neq Jx_A$,

iterations minimize the error

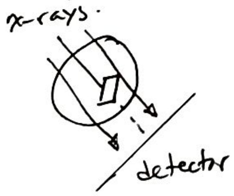
Back projection is $x = J^T m_A$

Filter is $J^TJ = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$ spatial low pass filter

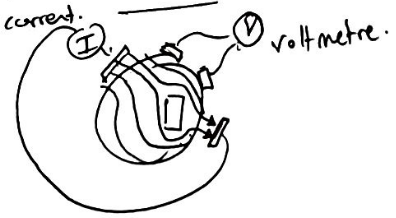
$(J^TJ)^{-1} \simeq \frac{1}{10}\begin{bmatrix} 4 & -1 & -1 & -1 \\ -1 & 4 & -1 & -1 \\ -1 & -1 & 4 & -1 \\ -1 & -1 & -1 & 4 \end{bmatrix}$ spatial high pass filter

☆ this is type #2 filtered back projection.

CT



EIT



☆ We made it, on to the pictures!

☆ density

☆ conductivity.

- individual x-rays go in a straight line.

- current flows everywhere
- small changes in measurements can indicate large changes in conductivity

— measurement noise —

for EIT

$$x_{n+1} = x_n + (J^T \underbrace{W}_{\text{inverse measurement covariance}} J + \lambda^2 \underbrace{R}_{\text{regularization}})^{-1}(J^TWb_n + \lambda^2 R(\underbrace{x_*}_{\text{prior estimate}} - x_n))$$