Propagation of Measurement Noise into Images

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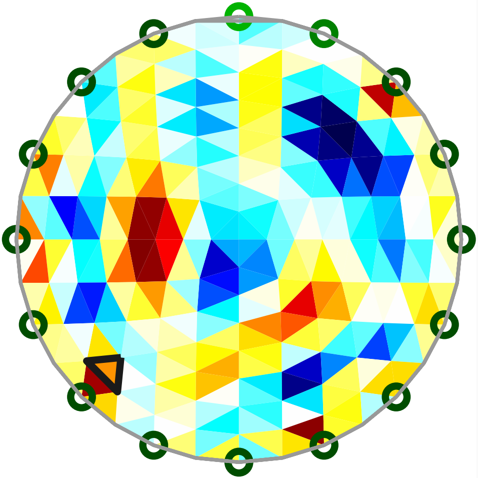
We Choose One Image



λ_L = 5.9e-02; λ_{GCV} = 3.7e-02

Measurements are Reconstructed

Single step Gauss-Newton reconstruction

$$\mathbf{x} = (\mathbf{J}^\intercal \mathbf{W} \mathbf{J} + \lambda^2 \mathbf{R})^{-1} \mathbf{J}^\intercal \mathbf{W} \mathbf{b}$$

x: change in conductivity; **b**: difference measurements;
J: Jacobian; **W**: inverse noise covariance;
λ: hyperparameter; **R**: regularization

Measurements are Reconstructed

Single step Gauss-Newton reconstruction

$$\mathbf{x} = \mathbf{Q}\mathbf{b}$$

x: change in conductivity; **b**: difference measurements;
$\mathbf{Q} = (\mathbf{J}^\mathsf{T}\mathbf{W}\mathbf{J} + \lambda^2\mathbf{R})^{-1}\mathbf{J}^\mathsf{T}\mathbf{W}$: reconstruction matrix

Noise is Reconstructed

$$\mathbf{x} = \mathbf{Q}(\mathbf{b} + \eta)$$

x: change in conductivity; **b**: difference measurements;
Q: reconstruction matrix;

η: noise

Additive noise, but of no particular distribution.

Noise is Reconstructed

$$\mathbf{x} = \mathbf{Qb} + \color{red}{\mathbf{Q}\eta}$$

x: change in conductivity; **b**: difference measurements; **Q**: reconstruction matrix;

η: noise

Noise is not Normal

but if it was...

Sample from the noise distribution, whether normal or otherwise, and linearly combine with other measurement noise samples using **Q**.

Noise is not Normal

but if it was...

These are *sums of random variables* (not a mixture distribution). If we have $\mu = 0$ (captured in **b**) then we can combine them as a *sum of weighted variances*.

$cx_1 \sim \mathcal{N}(c\mu, c^2\sigma^2)$; $\quad x_1 + x_2 \sim \mathcal{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ for indep x_1, x_2

Use a row of **Q** to scale and add measurement distributions into a conductivity distribution/uncertainty on a single voxel of **x**.

Central Limit and Bootstrap

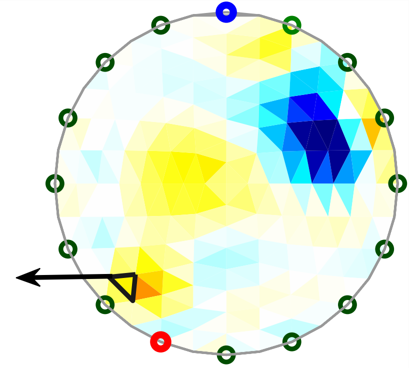
Empirical methods for estimating image noise are *bootstrap* or *leave one out*.

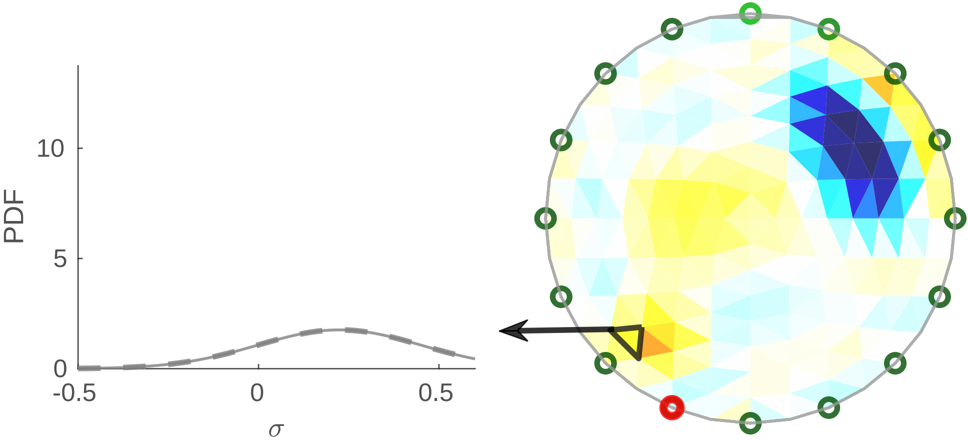
Estimates of the mean are guaranteed to approach a Gaussian distribution, with sufficient sampling, due to the *Central Limit Theorem*.
...but we can also just calculate the distribution directly, given our linear reconstruction matrix **Q** and a parametric noise distribution η when they're available.

SIMULATIONS

Simulation

- 2D, 16 electrodes, time difference **b**
- ◁ observe distribution of 1 voxel
- measurement noise $\eta \sim \mathcal{N}(0, c/10)$ for $c = \mathrm{var}(\mathbf{b})$
- electrode#10 noise $\eta_{10} \sim \mathcal{N}(0, c)$
- removed stimulus or measurements using electrode#*k*





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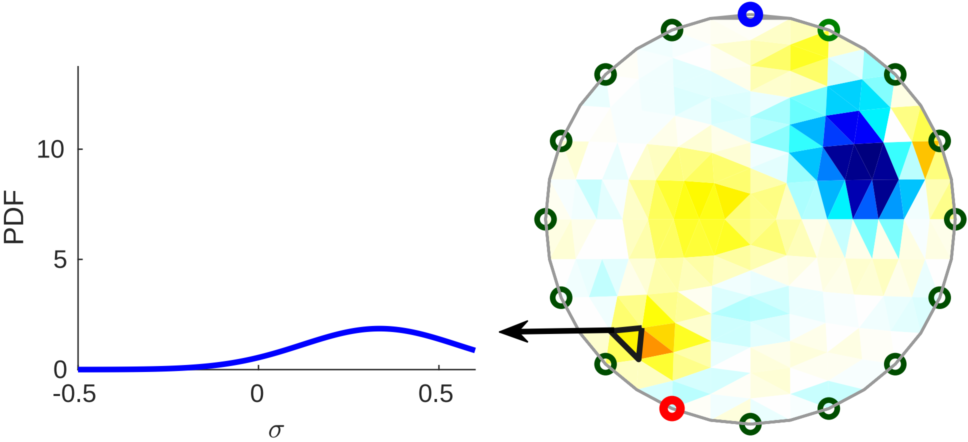

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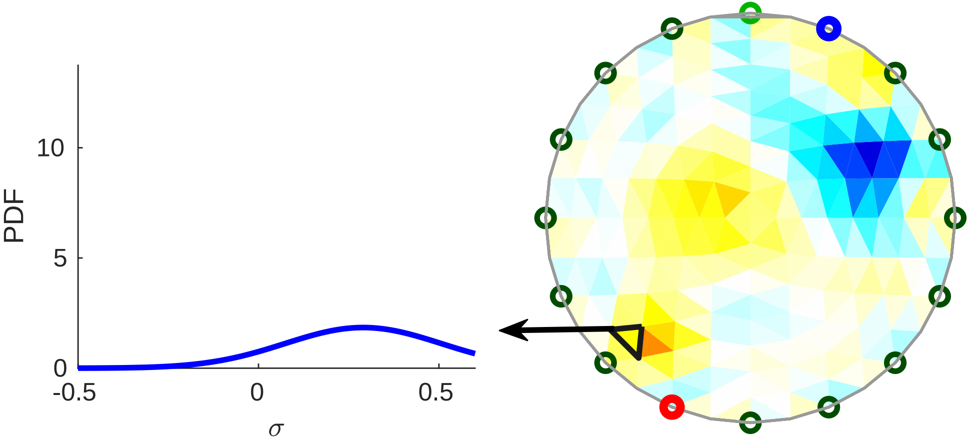

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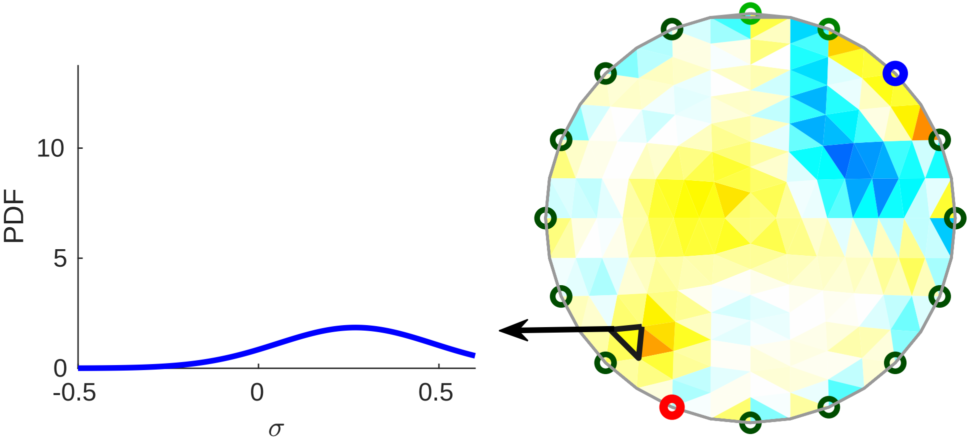

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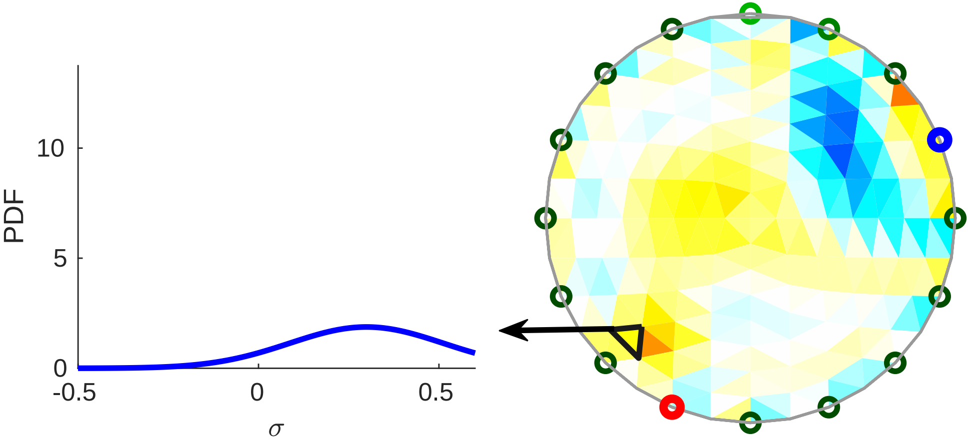

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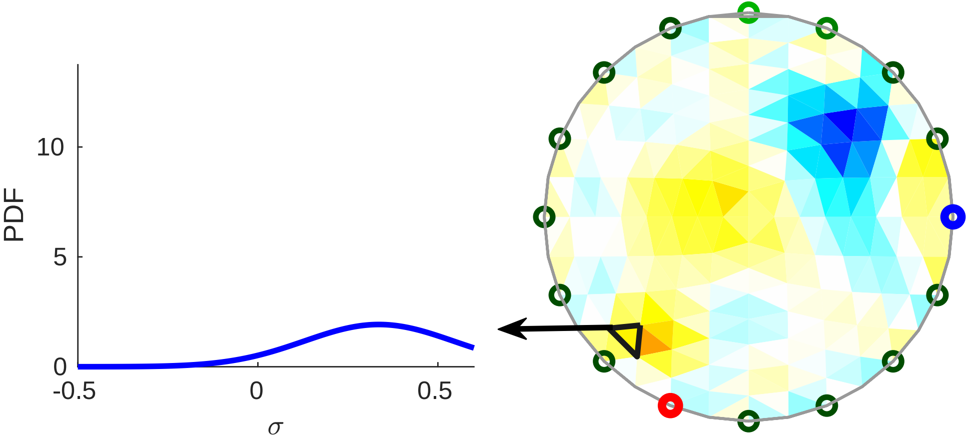

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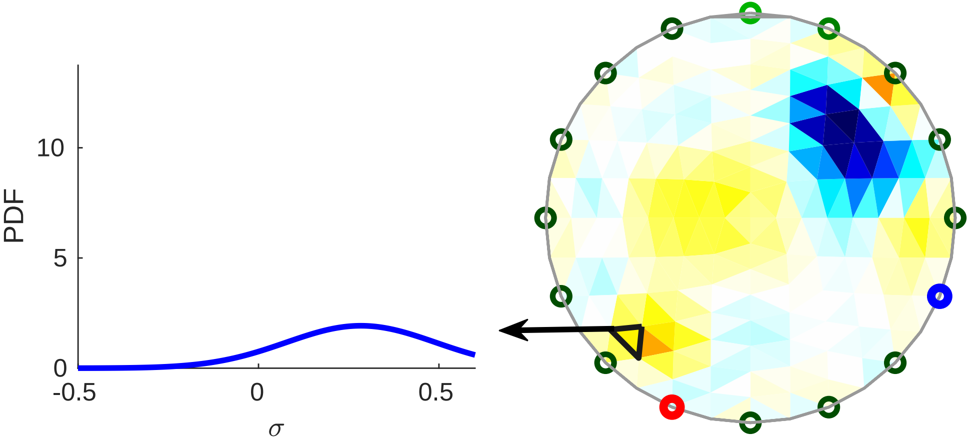

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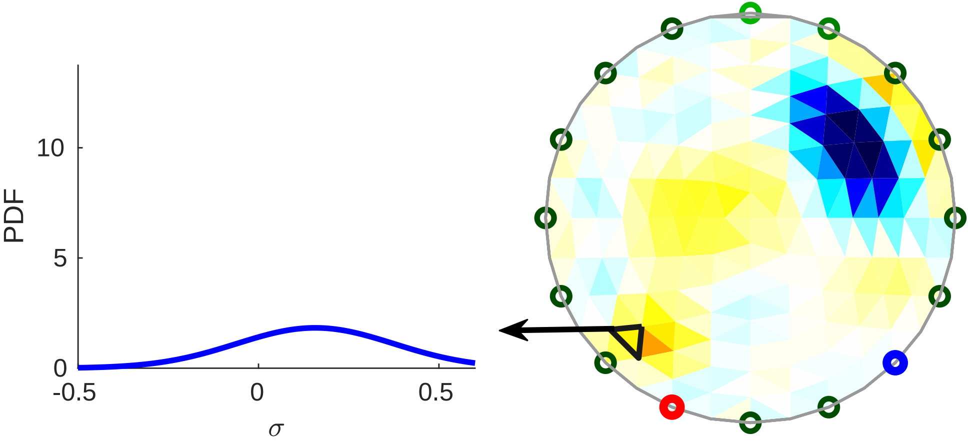
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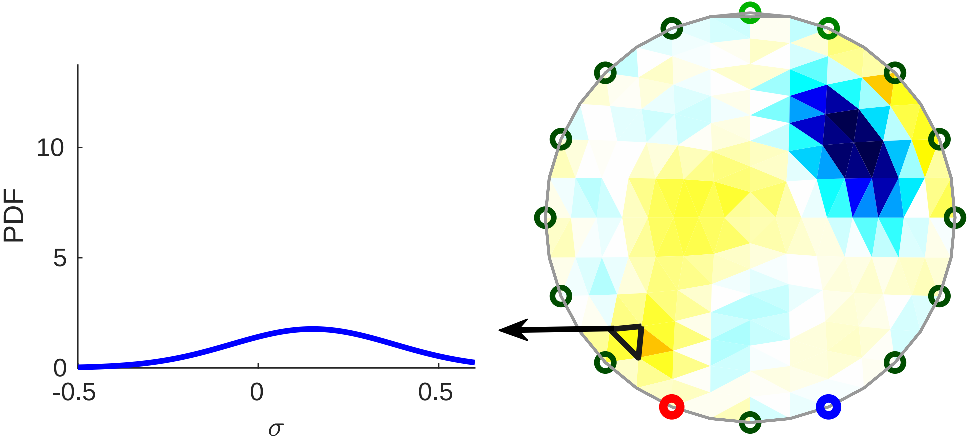

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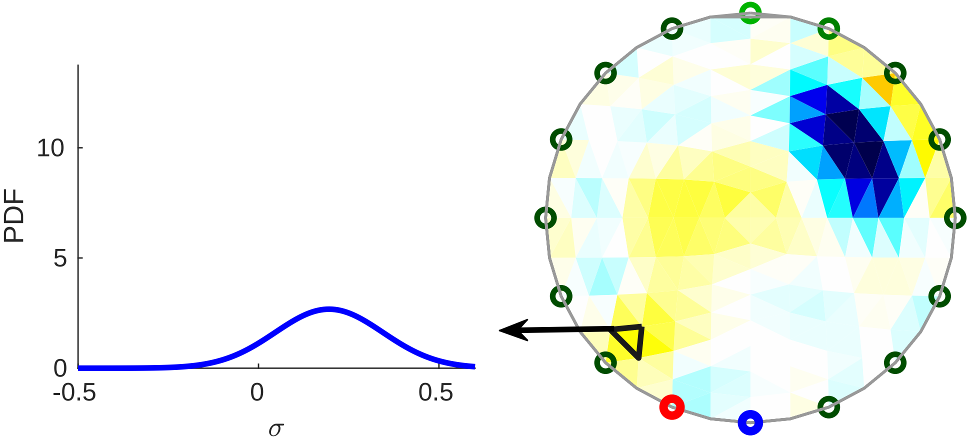
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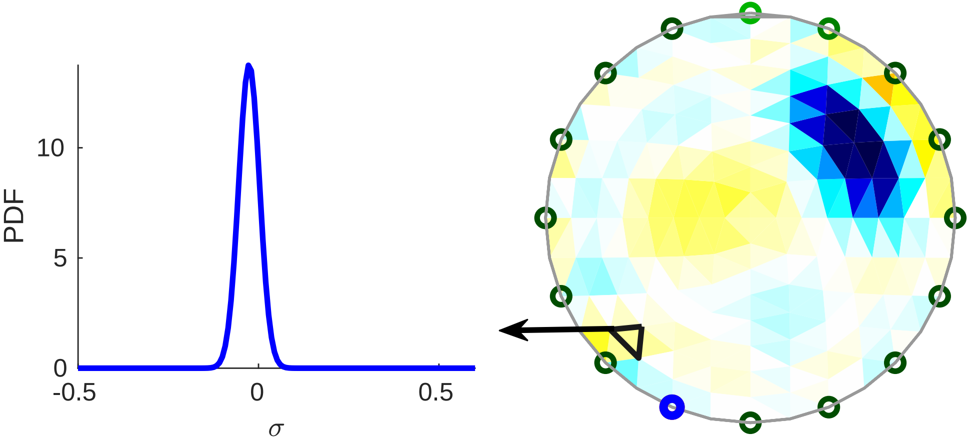
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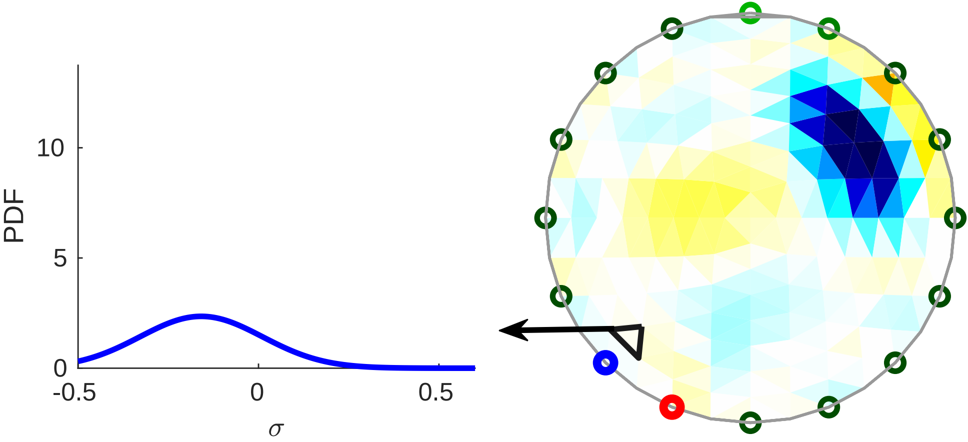
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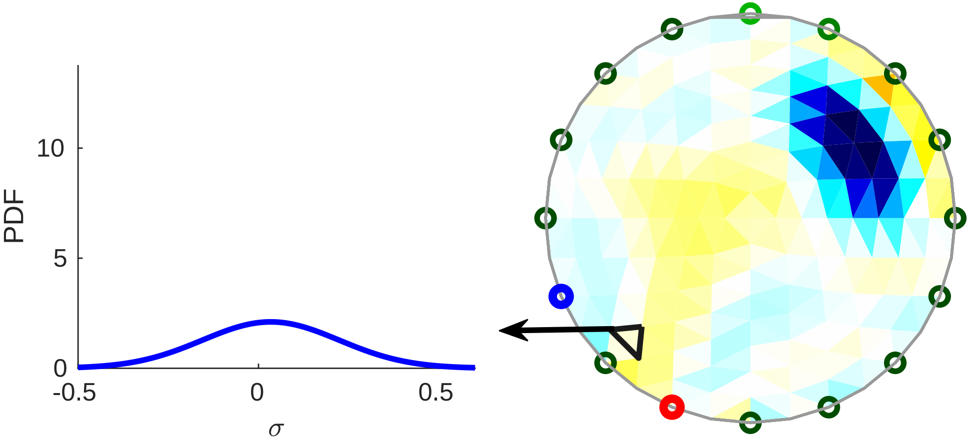

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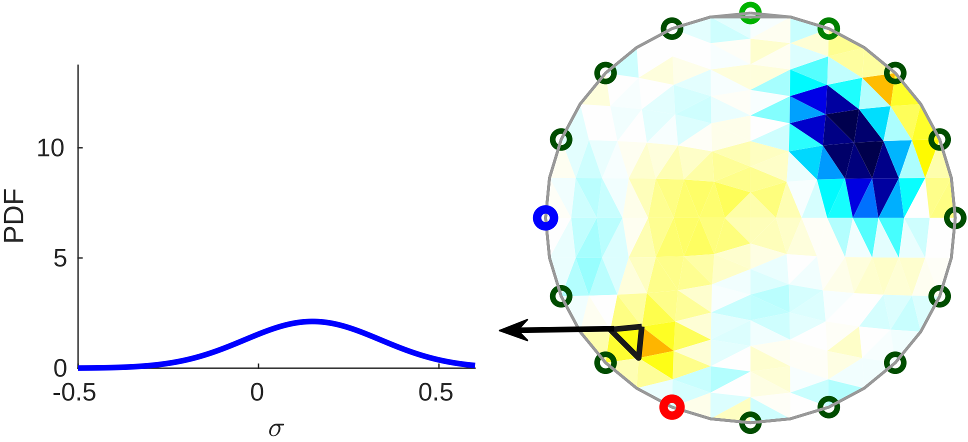
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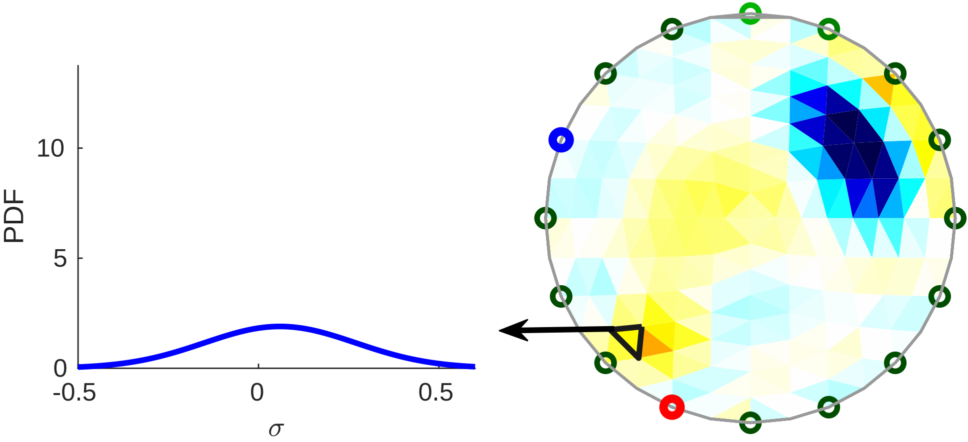
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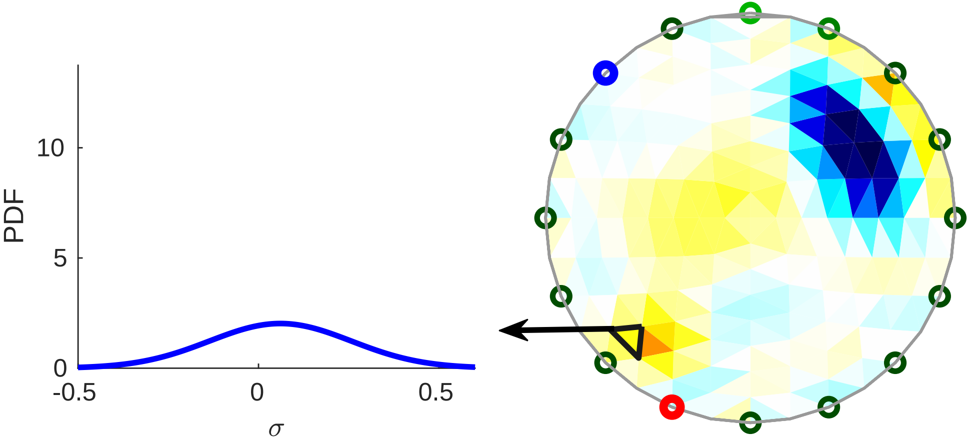
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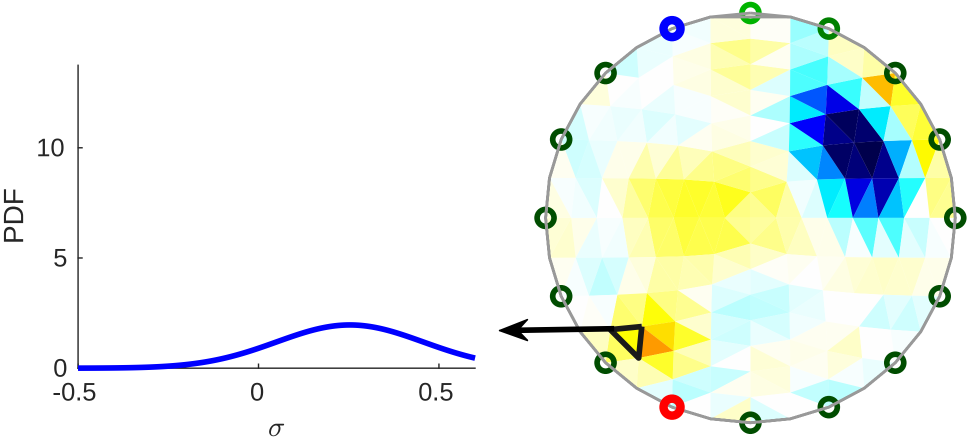

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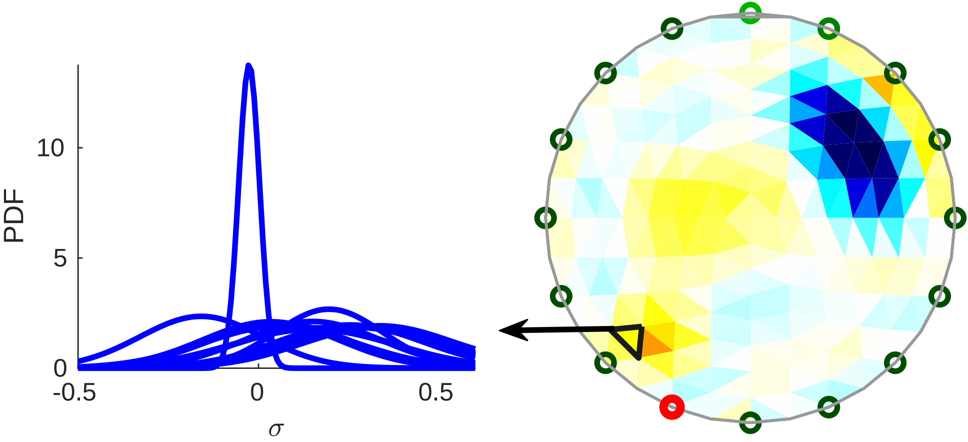
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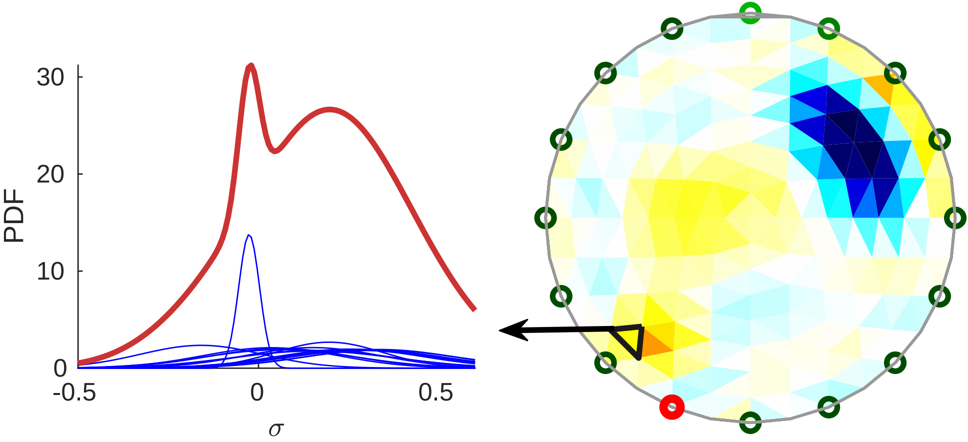
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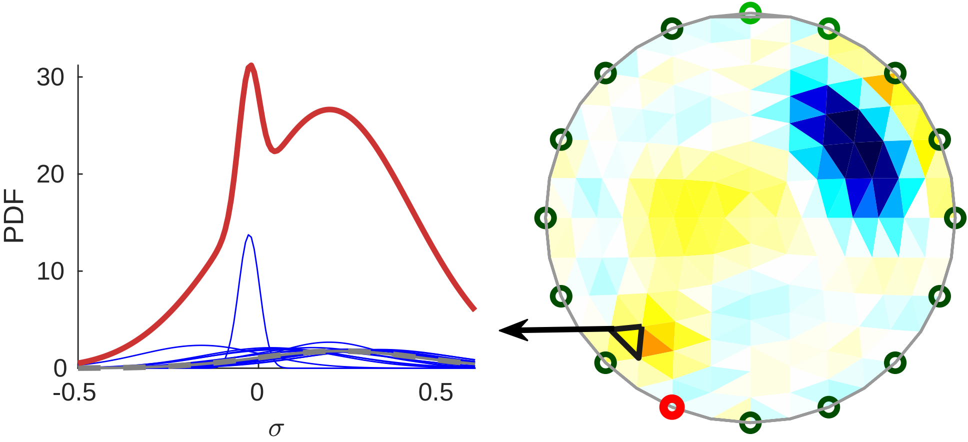

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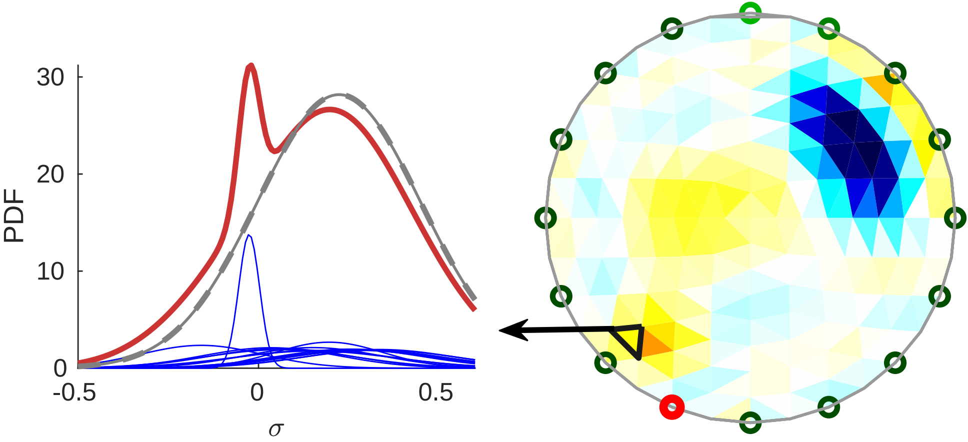

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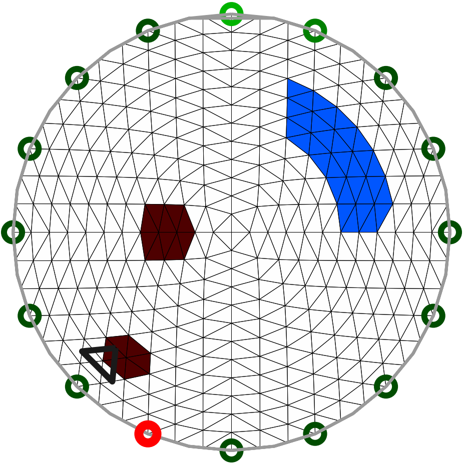
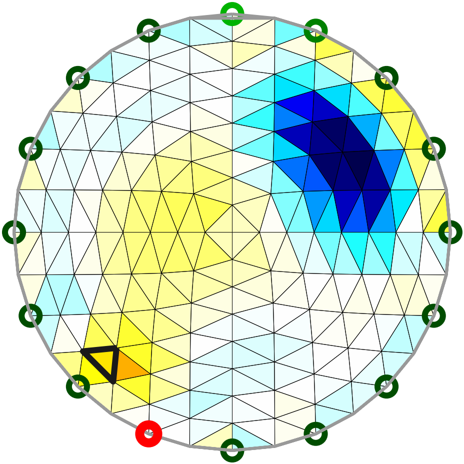
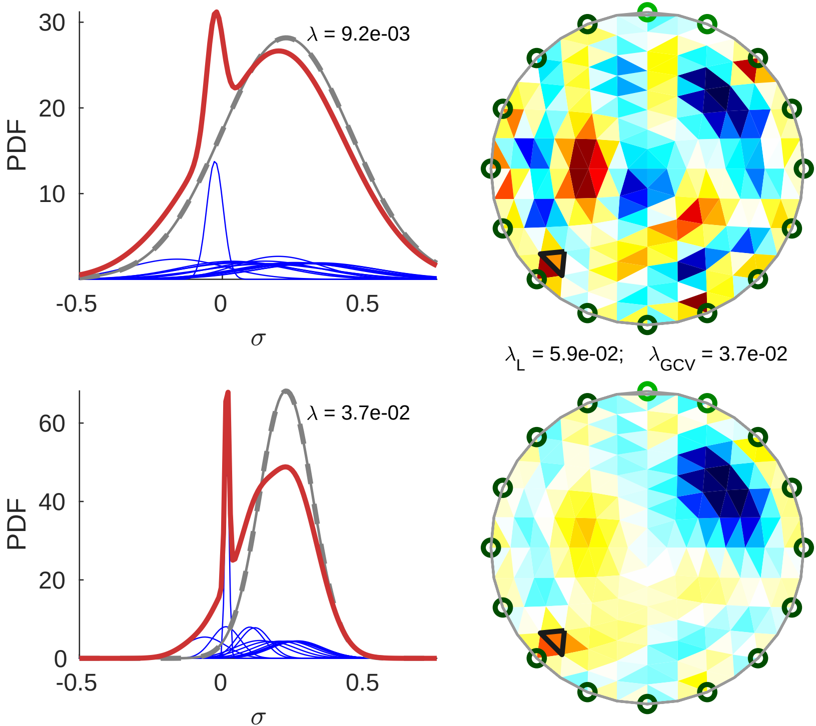
OBSERVATIONS

What can we observe?

- λ_{GCV} tests leaving out a single measurement...doesn't help here
- drop large variance measurements...
- offers a method for efficient calculation of simulated noisy images
- regularization suppresses the effect of measurement variance



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λ = 9.2e-03
λ = 3.7e-02
λ_L = 5.9e-02; λ_GCV = 3.7e-02
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